Jordan-Wigner transformation in a higher-dimensional lattice

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A type of Jordan-Wigner transformation is constructed for a spin-one-half quantum system on a higher-dimensional lattice. The properties of the transformation are discussed briefly. No flux is attached to spinless fermions in the present case.

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The one-dimensional Jordan-Wigner (JW) transformation [1] has provided remarkable applications in condensed matter physics [2,3]. Recently, much effort has been devoted to constructing higher-dimensional JW transformations [4,5,6]. However, a naive JW transformation for a three-dimensional lattice has not been reported up to now. In this paper, I construct explicitly a kind of JW transformation which is different from previous ones in two dimensions, and which can easily be generalized to higher dimensions.

The JW transformation maps local spin-one-half operators $S^{\pm},S^{z}$, to fermionic operators $c,c^{\dagger}$:

$$S^{-}(\mathbf{x})=U(\mathbf{x})c(\mathbf{x}),\quad S^{\dagger}(\mathbf{x})=c^{\dagger}(\mathbf{x})U^{\dagger}(\mathbf{x}) , \tag{1}$$

where $U(\mathbf{x})$ is a nonlocal function of $c^{\dagger}c$ and is usually written as

$$U(\mathbf{x})=e^{i\pi\sum_{\mathbf{z}}w(\mathbf{x},\mathbf{z})c^{\dagger}(\mathbf{z})c(\mathbf{z})}. \tag{2}$$

It can be shown that $w(\mathbf{x},\mathbf{z})$ must satisfy the asymmetrical condition

$$e^{i\pi w(\mathbf{x},\mathbf{z})}=-e^{i\pi w(\mathbf{z},\mathbf{x})}\quad(\mathbf{x}\neq\mathbf{z}) , \tag{3}$$

in order for $c,c^{\dagger}$ to obey Fermi statistics. In a one-dimensional (1D) lattice, the simplest solution of Eq. (3) takes the form

$$w(x,z)=\theta(x-z) , \tag{4}$$

where $\theta(x)$ is the step function. Fradkin [4] and Want [5] have given the following solution in 2D:

$$w(\mathbf{x},\mathbf{z})=\arg(\mathbf{x}-\mathbf{z}) , \tag{5}$$

where the function $\arg(\mathbf{x})$ is the angle between $\mathbf{x}$ and an arbitrarily given direction.

Here, I find a solution for Eq. (3) in 2D:

$$w(\mathbf{x},\mathbf{z})=\theta(x_1-z_1)(1-\delta_{x_1z_1})+\theta(x_2-z_2)\delta_{x_1z_1} , \tag{6}$$

where x_1,x_2 and z_1,z_2, respectively, are coordinate components of $\mathbf{x}$ and $\mathbf{z}$ under a given frame $(\mathbf{e}_1,\mathbf{e}_2)$. δ_{xz} is a one-dimensional lattice delta function. The generalization to higher dimensions is straightforward. For example, in 3D we have

$$\begin{aligned}w(\mathbf{x},\mathbf{z})=&\theta(x_1-z_1)(1-\delta_{x_1z_1})\\&+\theta(x_2-z_2)\delta_{x_1z_1}(1-\delta_{x_2z_2})\\&+\theta(x_3-z_3)\delta_{x_1z_1}\delta_{x_2z_2} .\end{aligned} \tag{7}$$

Now let us discuss the properties of the transformational function $w(\mathbf{x},\mathbf{z})$ proposed here. For simplicity, I focus only on the two-dimensional case. However, most of the conclusions given below are independent of dimension. $w(\mathbf{x},\mathbf{z})$ may be viewed as a Green's function related to a spinless fermion located at position $\mathbf{z}$. In analogy to the 1D case, only two possible values 1 and 0 can be taken by w. w equals zero at the left of the fermion, and equals 1 at the right. On line $x_1=z_1$, which crosses through the fermion, w has a jump. Similarly, a singular line (branch cut [4]) also exists for expression (5). However, the former divides a lattice plane into two separated parts, while the latter one does not. This is a significant difference because there is a different topology related to each one. As a result, it is easy to verify by using Eq. (6) that

$$\Delta w=0 \tag{8}$$

for the total change of w on any closed loop.

When the JW transformation has been performed, a phase factor which reads

$$e^{i\pi\int_{\mathbf{x}}^{\mathbf{y}}\mathbf{A}\cdot ds} \tag{9}$$

appears at the middle of two fermionic operators $c^{\dagger}(\mathbf{x}),c(\mathbf{y})$ in the final Hamiltonian, where the lattice line integration is defined in the same way as a continuum one, and bosonic field operator $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{x})=\sum_{\mathbf{z}}\mathbf{A}(\mathbf{x},\mathbf{z})c^{\dagger}(\mathbf{z})c(\mathbf{z}).$$

Vector $\mathbf{A}(\mathbf{x},\mathbf{z})$ is the lattice gradient of $w(\mathbf{x},\mathbf{z})$:

$$\begin{aligned}&A_j(\mathbf{x},\mathbf{z})=\nabla_j w(\mathbf{x},\mathbf{z}) ,\\&\nabla_j w(\mathbf{x},\mathbf{z})\equiv w(\mathbf{x}+\mathbf{e}_j,\mathbf{z})-w(\mathbf{x},\mathbf{z}) .\end{aligned} \tag{10}$$

$\mathbf{A}(\mathbf{x},\mathbf{z})$ has been interpreted as a gauge field [4] produced

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by a fermion located at position **z**. Substituting Eq. (6) into Eq. (10), it is not difficult to obtain an explicit expression for $\mathbf{A}(\mathbf{x},\mathbf{z})$ in 2D:

$$A_1(\mathbf{x},\mathbf{z})=[1-\theta(x_2-z_2)]\delta_{x_1 z_1} +\theta(x_2-z_2)\delta_{x_1+1,z_1} ,$$

$$A_2(\mathbf{x},\mathbf{z})=[\theta(x_2+1-z_2)-\theta(x_2-z_2)]\delta_{x_1 z_1} .$$

w can be expressed in terms of **A**:

$$w(\mathbf{x},\mathbf{z})=w(\mathbf{x}_0,\mathbf{z})+\int_{\mathbf{x}_0}^{\mathbf{x}}\mathbf{A}\cdot\mathbf{ds} . \quad (11)$$

It is clear that Eq. (8) is equivalent to

$$\oint \mathbf{A}\cdot\mathbf{ds}=0 \quad (12)$$

or

$$\epsilon_{ij}\nabla_i A_j(\mathbf{x},\mathbf{z})=0 . \quad (13)$$

Equations (12) or (13) tell us that fluxes related to the gauge vector potential $\mathbf{A}_\mathbf{z}(\mathbf{x})\equiv\mathbf{A}(\mathbf{x},\mathbf{z})$ vanish in the present case. This result is quite different from previous ones [4,5] which show that a flux tube of one-half flux quantum is attached to the fermion. In spite of the zero flux, it is impossible to eliminate the phase factor by a gauge transformation which does not violate the commutation relations. In fact, the two identities

$$\int_\mathbf{x}^\mathbf{y}\mathbf{A}_\mathbf{z}\cdot ds+\int_\mathbf{y}^\mathbf{z}\mathbf{A}_\mathbf{x}\cdot ds+\int_\mathbf{z}^x\mathbf{A}_\mathbf{y}\cdot\mathbf{ds}=1 \pmod 2 , \quad (14)$$

$$\epsilon_{ij}\tilde{\nabla}_i A_j(\mathbf{x},\mathbf{z})|_{\mathbf{z}=\mathbf{x}}=1 \pmod 2 , \quad (15)$$

which come directly from Eq. (3), rule out such a possibility. Here operator $\tilde{\Delta}_i$ acts on the second variable **z** of $\mathbf{A}(\mathbf{x},\mathbf{z})$, i.e.,

$$\tilde{\nabla}_i A_j(\mathbf{x},\mathbf{z})=A_j(\mathbf{x},\mathbf{z}+\mathbf{e}_i)-A_j(\mathbf{x},\mathbf{z}) .$$

It is highly unusual that the gauge field has no flux but is not gauge trivial. The result may be a consequence of the infinite range of the gauge interaction introduced here, or may be a lattice effect. On the other hand, zero flux indicates that bosonic field **A** could not be strictly interpreted as a gauge potential, and that flux is no longer suitable to describe physics uniquely in the present case. Paying attention to the fact that identity (14) is satisfied by **A**, we expect there exists an "average flux," contributed from a group of trifermions, which may be a good candidate for a meaningfully physical variable. In any case, we can conclude that the flux shown previously [4,5] could be canceled at the price of introducing a highly nonlocal bosonic field.

I have given a kind of higher-dimensional JW transformation which maps spin operators into fermionic operators. In spite of its highly nonlocal characteristic, applications to a higher-dimensional Heisenberg model may be helpful to obtain deeper insight. This will be discussed later.

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